# A DATABASE OF GROUP ACTIONS ON RIEMANN SURFACES

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Abstract. The automorphism group of a Riemann surface is important in a number of different mathematical fields. An algorithm of Thomas Breuer provides ways to determine all such groups for a fixed genus, but data generated from this algorithm did not include the generators of the monodromy group, which are also valuable. This paper describes modifications the author made to Breuer's code to add the generators, as well as other new code to compute additional information about a given Riemann surface. Data from this project has been incorporated into the *L-functions and Modular Forms Database* (`http://www.lmfdb.org`) and we also describe the relevant data there.

## 1. Introduction

Groups acting on Riemann surfaces are important to a range of mathematical topics from Galois theory of extensions of $\mathbb{C}(z)$ [Völklein, 1996], to Jacobian variety decompositions [Paulhus, 2008], to Galois covers of the projective line corresponding to Shimura varieties [Frediani et al., 2015], to questions about indecomposable rational functions [Fried, 1973]. Most of these topics utilize the generators of the monodromy group of the covering corresponding to the mapping $X \to X/G$ from a Riemann surface $X$ to the orbit space of $X$ by the group $G$ acting on it.

Breuer created computer code to determine all groups acting on Riemann surfaces of a given genus [Breuer, 2000]. He ran the code up to genus 48, and recorded the groups along with limited information about the ramification of the mapping $X \to X/G$. Within his code, generators of the monodromy group were also computed, but not recorded. We added functionality to Breuer's code to fully compute these generators, and wrote new code to compute additional information about Riemann surfaces. As this data will aid other researchers, we are creating a publicly visible, easily accessible database containing this data.

Enter the *L-functions and Modular Forms Database* (henceforth called "LMFDB"), a huge database of mathematical objects. As an established database with a strong infrastructure, LMFDB is an ideal location to post this data. Part of its goal is to provide opportunities for unexpected connections between mathematical concepts. This paper describes the modifications we made to Breuer's code, as well as additional computations (such as which actions correspond to full automorphism groups, and which correspond to hyperelliptic curves) we use to generate data on LMFDB. The relevant code may be found at `http://github.com/jenpaulhus` and the database is at `http://www.lmfdb.org/HigherGenus/C/Aut`.

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Section 2 is an overview of the necessary mathematical background on groups acting on Riemann surfaces, and in Section 3 we describe the theoretical underpinnings of the original code of Breuer. In Section 4 we explain the new pieces of mathematical information added to the data and discuss the organization of the data on LMFDB. Finally in Section 5 we enumerate planned future additions to the database.

## 2. Background on Riemann Surfaces

Let $X$ be a compact Riemann surface of genus $g \geq 2$ (also referred to as a "curve"), and let $G = \mathrm{Aut}(X)$, the group of biholomorphic maps from $X$ to itself. It is well known that this group is finite and bounded in size by $84(g-1)$. There is a natural mapping $\phi : X \to Y = X/G$ where $Y$ is the orbit space of $X$ under the action of $G$ ($\phi$ sends $x \in X$ to the orbit of $x$ under the action of $G$), and $g_0$ denotes the genus of the quotient $Y$. It is possible that this mapping branches at several points of $Y$, say on a set $\mathcal{B} \subset Y$ of size $r$. Letting $\phi^{-1}(\mathcal{B}) \subset X$ be the inverse image of these points, the mapping from $X - \phi^{-1}(\mathcal{B})$ to $Y - \mathcal{B}$ is a degree $d$ covering for some positive integer $d$. For details on the covering space theory used in the paper, we recommend [Lee, 2011, Chapters 11 and 12]. For our specific situation, we recommend [Fried, 1980] or [Breuer, 2000].

Fix a base point $y_0 \in Y - \mathcal{B}$. Then $\phi^{-1}(y_0)$ consists of $d$ points in $X - \phi^{-1}(\mathcal{B})$, say $\phi^{-1}(y_0) = \{x_1, \ldots, x_d\} \subset X$. Now consider a loop starting at $y_0$ and traveling once around one branch point in $\mathcal{B}$. For each element $x_i$ in $\phi^{-1}(y_0)$ this loop lifts uniquely to a path in $X$ which starts at $x_i$ and ends at some $x_j \in \phi^{-1}(y_0)$. This defines a permutation on the $d$ elements of $\phi^{-1}(y_0)$: send $i$ to the number of the endpoint of the corresponding lift starting at $x_i$. These $r$ permutations induce a map $\rho : \pi_1(Y - \mathcal{B}, y_0) \to S_d$ where $S_d$ is the symmetric group on $d$ elements, and the image of $\rho$ is called the *geometric monodromy group* which is isomorphic to $\mathrm{Aut}(X)$ in the case of Galois covers. The order of each permutation corresponding to a loop around one element of $\mathcal{B}$ is denoted $m_i$ for $1 \leq i \leq r$. When $X$ and $Y$ are connected, the image of $\rho$ is a transitive subgroup of $S_d$.

The universal cover of a compact Riemann surface is the upper half plane $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ which has automorphism group $\mathrm{PSL}(2, \mathbb{R})$, and so $X$ may be described as the orbit space of $\mathbb{H}$ by a torsion free subgroup of $\mathrm{Aut}(\mathbb{H})$ (see [Breuer, 2000, Theorem 3.9] or [Jones and Singerman, 1987, 4.19.8]). Call that torsion free subgroup $K$. It is isomorphic to $\pi_1(X, x_0)$.

Similarly, $Y$ is equivalent to the orbit space of $\mathbb{H}$ by a subgroup $\Gamma$ of $\mathrm{PSL}(2, \mathbb{R})$ called a *Fuchsian group*. These Fuchsian groups have an explicit presentation [Breuer, 2000, Theorem 3.2]:

$$\Gamma = \langle \alpha_1, \beta_1, \ldots, \alpha_{g_0}, \beta_{g_0}, \gamma_1, \ldots, \gamma_r \mid \prod_{i=1}^{g_0} [\alpha_i, \beta_i] \prod_{j=1}^{r} \gamma_j = 1, \gamma_j^{m_j} = 1 \rangle \tag{1}$$

where $[\alpha_i, \beta_i]$ is the commutator of $\alpha_i$ and $\beta_i$. The list of non-negative integers $[g_0; m_1, \ldots, m_r]$ is called the *signature* of $\Gamma$ and is uniquely determined for each Fuchsian group. The action of $\Gamma$ on $\mathbb{H}$ induces an action of $\Gamma/K$ on $\mathbb{H}/K$, so $G \cong \Gamma/K$. As such we have an exact sequence

$$1 \to K \xrightarrow{\iota} \Gamma \xrightarrow{\eta} G \to 1. \tag{2}$$

Then, $G = \mathrm{Aut}(X)$ may also be defined as the image of a *surface kernel epimorphism*, a surjection $\eta : \Gamma \to G$. Observe that different surface kernel epimorphisms may exist for fixed groups $\Gamma$ and $G$. So to classify actions it is not sufficient to only give the group and signature. We also need to describe the map $\eta$ via, say, a description of where $\eta$ sends the generators. Due to the structure of $\Gamma$, the group $G$ may be completely defined by $2g_0$ *hyperbolic* generators $a_1, b_1, \ldots, a_{g_0}, b_{g_0}$ and $r$ *elliptic* generators $c_1, \ldots, c_r$ such that the $c_i$ have order $m_i$ and the product $\prod_{i=1}^{g_0}[a_i, b_i]\prod_{j=1}^{r} c_j = 1_G$ where $1_G$ is the identity element of $G$. We call this list of $2g_0 + r$ generators of $G$ a *generating vector*.

Conversely, suppose $G$ is any transitive subgroup of some symmetric group $S_d$ with $2g_0 + r$ generators $\{a_1, b_1, \ldots, a_{g_0}, b_{g_0}, c_1, \ldots, c_r\}$ such that the $c_i$ have order $m_i$ and $\prod_{i=1}^{g_0}[a_i, b_i]\prod_{j=1}^{r} c_j = 1_G$. We say such a group has *product one generators*, and a set of $2g_0 + r$ generators is a *product one generator*. Then any surjection $\eta : \Gamma \to G$ defined as $\eta(\alpha_i) = a_i$, $\eta(\beta_i) = b_i$, and $\eta(\gamma_i) = c_i$ has a corresponding kernel $K$, and $G$ acts on the compact Riemann surface $X$ defined as the orbits of $K$ acting on $\mathbb{H}$.

Hence there is a one-to-one correspondence between surjective maps $\eta : \Gamma \to G$ with $\ker(\eta)$ a torsion free group and finite groups which have product one generators. This is the beautiful existence theorem of Riemann (really a generalization of it) and it gives a way to translate the topological language of ramified coverings to the world of generators of finite groups. There are several very good sources on Riemann's existence theorem, particularly [Fried, 1980]. For a brief survey with generalizations and historical perspectives, see [Harbater, 2015]. The topic is also treated briefly in [Miranda, 1995, pg. 90-94], or in relation to function fields and the Inverse Galois Problem [Völklein, 1996].

As with most mathematical objects, many unequal surface kernel epimorphisms exhibit identical behaviors. For example, relabeling the elements of $\phi^{-1}(y_0)$ (or reordering the $c_i$) should not constitute creating a "new" action. There are a number of different equivalence relations that may be placed on the surface kernel epimorphisms and we must make choices about which equivalence relation to classifying group actions up to in the database. For this work, we consider an equivalence relation which is slightly weaker than topologically or analytically equivalent, meaning two distinct group actions in our database may actually be topologically (or even analytically) equivalent. In Section 5 we discuss the idea of analytically and topologically equivalent actions in relation to future work.

Let $G$ be a finite group which is the image of a surface kernel epimorphism $\eta : \Gamma \to G$, with $[g_0; m_1, \ldots, m_r]$ the signature of $\Gamma$. We denote by $\mathcal{C} = (C_1, \ldots, C_r)$ a list of $r$ conjugacy classes in $G$ (not necessarily distinct) each containing elements of order $m_i$. Define $S$ to be the set $\{(s_1, \ldots, s_r) : s_i \in C_i\}$. Then $G$ acts on $S$ by component-wise conjugation. We denote conjugation of $s_i$ by some $h \in G$ as $s_i^h$ and refer to this action as *simultaneous conjugation.*

In the special case when these tuples are generating vectors, any two vectors in the same orbit under simultaneous conjugation represent conformally equivalent actions in the Riemann surface (although the converse is not always true). This follows from the definition of conformal equivalence (see Section 5.1) and the fact that conjugation is an element of $\mathrm{Aut}(G)$. Notice that the properties of a tuple in $S$ being a product one generator are invariant under simultaneous conjugation. This equivalence relation is the one we use to classify our actions.

Given a Riemann surface $X$ of genus $g$, a group $G$ acting on $X$, a tuple $\mathcal{C} = (C_1, \ldots, C_r)$ of conjugacy classes of $G$, and a generating vector $(s_1, \ldots, s_r)$ with $s_i$ in $C_i$, then the tuple $(g, G, \mathcal{C})$ is called a *refined passport* [Sijsling and Voight, 2014] (alternatively that $X$ *is of ramification type* $(g, G, \mathcal{C})$ [Magaard et al., 2002]). A *passport* is a similar tuple of information, but the conjugacy classes are only considered in $S_d$, so the actions are only classified up to the cycle type of the generators of $G$.

## 3. Breuer's Code

Breuer's contribution to this topic was to devise an algorithm to generate a list of all groups and corresponding signatures for which there is a surface kernel epimorphism $\eta : \Gamma \to G$ for a fixed genus. We only give a brief overview of his algorithm here (see [Breuer, 2000] for more details).

Breuer's algorithm first generates a list of all possible signatures for Fuchsian groups $\Gamma$ for a given genus $g$ and given order $n$ of the automorphism group, using combinatorial restrictions on possible $m_i$ values, as well as the Riemann-Hurwitz formula.

Next the algorithm searches the small group database in [GAP, 2006] and uses group theoretic results to construct a list of groups $G$ of order $n$ which could have one of the determined admissible signatures for that $n$. If a group of order $n$ does not have elements of orders corresponding to the values in the signature, it is removed from the list of potential automorphism groups.

Finally, the algorithm determines which possible groups $G$ satisfy the condition that there is a surjective morphism $\eta : \Gamma \to G$. This step in the algorithm utilizes several different group theoretic results concerning the structure of conjugacy classes. The algorithm first determines all possible lists of conjugacy classes $\mathcal{C} = (C_1, \ldots, C_r)$ such that the order of elements in $C_i$ is $m_i$ (so potential refined passports for a given genus and group). Then a result in [Scott, 1977, Theorem 1] gives a sufficient condition on the irreducible characters of a group $G$ to show there is not a surjective homomorphism $\eta : \Gamma \to G$. Similarly, Breuer computes the size of $\mathrm{Hom}_{\mathcal{C}}(g_0, G)$, the set of homomorphisms from the Fuchsian group corresponding to the given signature to the group $G$, using the following theorem.

**Theorem 3.1** (Theorem 3, [Jones, 1995]). *With $\mathcal{C} = (C_1, \ldots, C_r)$ as above,*

$$|Hom_{\mathcal{C}}(g_0, G)| = |G|^{2g_0-1} \sum_{\chi \in Irr(G)} \chi(1)^{2-2g-r} \prod_{i=1}^{r} \sum_{\sigma_i \in C_i} \chi(\sigma_i).$$

When this value is 0, there cannot be a surface kernel epimorphism for that refined passport.

Conversely, to show there is an epimorphism $\eta : \Gamma \to G$, a specific generating vector defining the particular surface kernel epimorphism must be found (as the images in $G$ of $\alpha_i, \beta_i, \gamma_j$ from (1) under the mapping $\eta$). A brute search of all possible generating vectors for a given refined passport is not feasible, especially for large signatures or large groups.

Instead Breuer uses the following proposition to quickly generate one element of each orbit under the action of simultaneous conjugation.

**Proposition 3.2** (Lemma 15.27, [Breuer, 2000]). *Fix elements $\sigma_i \in C_i$ for each $1 \leq i \leq r$. Then the following set $T$ gives us precisely one representative for each orbit of the action of $G$ on $S = \{(s_1, \ldots, s_r) : s_i \in C_i\}$ by simultaneous conjugation:*

$$T = \{(\sigma_1, \sigma_2^{b_2}, \ldots, \sigma_r^{b_r}) : b_i \in R(b_1, \ldots, b_{i-1}) \text{ for } 2 \leq i \leq r\}$$

*where $R(b_1, \ldots, b_{i-1})$ is a set of representatives of the double coset*

$$C_G(\sigma_i)\backslash G / C_G(\sigma_1, \sigma_2^{b_2}, \ldots, \sigma_{i-1}^{b_{i-1}}),$$

*defined iteratively and where $C_G(g_1, g_2, \ldots, g_k)$ means the intersection of the centralizers of $g_i \in G$ for $1 \leq i \leq k$.*

Each element of $T$ is tested to see if it is a product one generator. Breuer did not record these generating vectors in his original data, though. His goal was to list group and signature pairs only.

## 4. New additions

As mentioned above, to fully classify group actions on Riemann surfaces, we need to know the generating vector for each action. We converted Breuer's code to the computer algebra language Magma [Bosma et al., 1997] to align the programs with other code written by the author. We also added functionality which, given a group and signature, outputs the generating vector(s) for each refined passport up to simultaneous conjugation generated via Proposition 3.2 (see [Paulhus, 2015], specifically the file `genvectors.mag`). This is the key piece of code, as we do not need to reproduce all of Breuer's program. We use his group and signature pairs as a starting point, and then add the generating vectors using the modified version of his code.

There is also a software package in GAP called `MapClass`, which, among other computations, finds the generating vectors given a group and list of conjugacy classes corresponding to a refined passport [James et al., 2012].

One important piece of information which is not determined in Breuer's original code is whether the group action described is the full automorphism group for the family of curves with corresponding data. Suppose we have an exact sequence

$$1 \to K \xrightarrow{\iota} \Gamma \xrightarrow{\eta} G \to 1$$

as in (2), and a corresponding generating vector from our modified version of Breuer's code. It is possible that there is some group $H$, Fuchsian group $\Gamma_0$ so that $G < H$, a mapping $j : \Gamma \to \Gamma_0$, and an exact sequence

$$1 \to K \xrightarrow{\iota_o} \Gamma_0 \xrightarrow{\eta_0} H \to 1$$

so that $\eta = \eta_0 \circ j$. In this case, the generic element of this family of Riemann surfaces has automorphism group $H$ and signature that of $\Gamma_0$.

In [Ries, 1993] there are conditions for determining exactly when this situation occurs. Given $G$ and $\Gamma$, the paper also describes explicitly how to compute $H$ and $\Gamma_0$. The cases $G \lhd H$ are covered in [Ries, 1993, Theorem pg. 390], while the remaining cases are covered in Table 1 and Table 2 of that paper. First, the signature of $\Gamma$ must match one of only a handful of signatures for which this scenario can happen. For example, if $g_0 = 0$ and there are more than 4 branch points, the given group $G$ is always the full automorphism group of the generic point of the family ($\eta$ in this case never satisfies the conditions outlined in [Ries, 1993]). If the signature is one of the few that might lead to a larger automorphism group, in the cases where $G \lhd H$, there must also exist an element of the automorphism group of $G$ that behaves in a certain way on the generating vector corresponding to this action $\eta$.

We have written code [Paulhus, 2016] which takes the output of the modified Breuer program and determines if the mapping $\eta$ defined by a generating vector satisfies one of the conditions outlined in Ries. When such an example is found, the group $H$ and signature of $\Gamma_0$ are also recorded. One caveat: the code only determines the group $H$ and signature of $\Gamma_0$, it does not determine exactly which refined passport (if there is more than one) the original group $G$ and signature correspond to. This should be possible to determine using information in the proof of Theorem pg. 390 in [Ries, 1993].

In the special case when the signature of the action is $[0; k, k, k]$ or $[0; k, k, k, k]$, we must determine if there exists an automorphism of $G$ which acts in a certain way on a generating vector *up to applying an element of* $Aut^+(\Gamma)$ *to the elements of the generating vector*, where $\mathrm{Aut}^+(\Gamma)$ is orientation preserving automorphisms of $\Gamma$.

In the two cases when this happens, $g_0 = 0$ so the group $\mathrm{Aut}^+(\Gamma)$ is the Artin braid group. This group is an infinite (but finitely generated) group generated by $Q_1, \ldots, Q_{r-1}$ where $Q_i$ is the mapping sending one generating vector $(s_1, s_2, \ldots, s_r)$ to $(s_1, \ldots, s_{i-1}, s_{i+1}, s_{i+1}^{-1} s_i s_{i+1}, s_{i+2}, \ldots, s_r)$ [Magnus et al., 1966, Section 3.7]. However, the orbit of a given generating vector under the action of the elements of the braid group is finite (since the group $G$ is finite there are only a finite number of generating vectors). To exhaustively determine whether the action corresponds to the full group, we need to generate the whole orbit of a given generating vector and test if there is an element of $\mathrm{Aut}(G)$ which acts on one of the generating vectors in that orbit in such a way to satisfy the conditions as described in Ries's paper.

To do this, given a generating vector and all cycles of it (or permutations if the group is abelian), we apply the braids $Q_1$, $Q_2$ (and $Q_3$ in the case of $[0; k, k, k, k]$) to the list of generating vectors and test all of the elements in this list against the condition set out in [Ries, 1993, Theorem pg. 390]. If we find an automorphism satisfying the conditions in this theorem, we have a candidate for the full automorphism group. If not, we apply the braids to the new larger set and repeat the process. This will eventually generate the whole orbit (if it doesn't find, along the way, a generating vector in the orbit which satisfies the condition mentioned above) and the program will terminate since the orbit is finite. If it terminates without finding a generating vector satisfying the conditions, the action represented by the initial generating vector must be the full automorphism group.

Once we determine whether an action represents the full automorphism group, we compute additional information connected to the given refined passports. An interesting property of the families of Riemann surfaces described by these actions is whether they are hyperelliptic curves or cyclic trigonal curves. A hyperelliptic curve of genus $g$ is defined by the presence in its automorphism group of a central involution with $2g + 2$ fixed points, while a cyclic trigonal curve of genus $g$ is defined by the presence of an automorphism of order 3 which fixes $g + 2$ points. Given a generating vector, code to compute the number of fixed points of a given automorphism (using [Breuer, 2000, Lemma 10.4]), and then determine if the curve is hyperelliptic or cyclic trigonal is in [Swinarski, 2016]. Code in that paper also computes the hyperelliptic involution or trigonal automorphism, which we include.

Once the generating vectors are known, two other key pieces of data about corresponding varieties are then computed. Work of the author gives a method to use the automorphism group of a curve (and the generating vectors of the action) to decompose its Jacobian variety [Paulhus, 2008]. The code to implement this method may be found at [Paulhus and Rojas, 2016]. An entry such as $E \times E^3 \times A_4 \times A_5^2$ in the database means the decomposition consists of four factors: an elliptic curve, three isogenous copies of (possibly) another elliptic curve, one dimension four abelian variety, and two isogenous copies of a dimension five abelian variety.

In the paper [Frediani et al., 2015], the authors define a way to compute the dimension of the corresponding Shimura variety, again given the generating vectors. They provide code to compute this dimension at `www.dima.unige.it/~penegini/publications/PossGruppigFix_v2Hwr.m`, which we run on all examples in the database.

Breuer's algorithm only asserts the existence of a family of Riemann surfaces with a particular group acting on it and with a particular signature. It is important to also know the equation(s) for the curves in this family. Determining an equation for a curve given an automorphism group and signature is, in general, a very hard problem. Equations are known for hyperelliptic curves [Shaska, 2003], genus 3 curves with automorphisms [Magaard et al., 2002], and genus 4-7 curves with "large" automorphism groups (the size of the automorphism group is at least $4(g-1)$) [Swinarski, 2016]. We added all these equations to the data with one small exception. In [Shaska, 2003] the equations are classified up to passports, not up to refined passports (the cycle structure of the generating vectors instead of the conjugacy classes in $G$). In two cases (if $G \cong C_2 \times C_2$, and if $G \cong C_4 \times C_2$ and the quotient of $G$ by the hyperelliptic involution is $C_2 \times C_2$) there is more than one equation listed in [Shaska, 2003] but in our data there are distinct refined passports which are in the same passport. The author does not know a way to determine which equation(s) correspond to which refined passport.

One note about our presentation of groups. Breuer's original code outputs a group as labeled in Magma or GAP, so as a pair $(a, b)$ which indicates the group is of order $a$ and is the $b$th group of that order in the database of small groups. Our Magma version of Breuer's code requires the group to be a permutation group to compute double coset representatives as in Proposition 3.2. However, in Magma, many groups of the form `SmallGroup(a,b)` are not permutation groups. Also, to correspond to the mapping $\rho : \pi_1(Y - \mathcal{B}, y_0) \to S_d$ from Section 2, the group $G$ must be transitive and satisfy the Riemann-Hurwitz formula. So we first convert the group to a permutation group, as in the standard proof of Cayley's theorem. The code to do this is at [Paulhus, 2016]. In doing so, we are specifying that our covers are Galois.

Putting this all together, the final process to create the database at `http://www.lmfdb.org/HigherGenus/C/Aut` is:

- For a fixed genus, load all the signature and group pairs computed with Breuer's original program and loop over this data.

- Convert groups of the form `SmallGroup(a,b)` in Breuer's data to permutation groups.
- Use our modified version of Breuer's code to determine the refined passports, and compute generating vector(s) for each.
- Determine if the action on each refined passport describes the full automorphism group of the family.
- Compute the Jacobian variety decomposition and the dimension of the corresponding Shimura variety.
- If the action is the full action, check if the family consists of hyperelliptic or cyclic trigonal curves. In special cases we add equations.
- Future additional information will be checked at this point.

4.1. **Organization of the data on LMFDB.** Currently the database contains data up to genus 15, and we have initially only included examples where the quotient $X_G$ is the Riemann sphere ($g_0 = 0$).

Each tuple of information: (genus, group, signature) has its own page on LMFDB. On each such page there is a list of the different refined passports corresponding to the given genus, group, and signature, and links to individual pages for each refined passport.

The individual pages of each refined passport list all generating vectors corresponding to this passport. We also list which conjugacy classes the refined passport corresponds to (as labeled by Magma when we initially generate the data). These pages also contain information about whether the action represents the full automorphism group of the family of Riemann surfaces. If the example is not the full automorphism group, a link to the action which does correspond to the full automorphism group is also included. We note if a refined passport of a full automorphism group corresponds to a hyperelliptic curve or a cyclic trigonal curve, and list the corresponding hyperelliptic involution or trigonal automorphism. Known equations are also displayed here.

On both types of pages, a download button is available which downloads a Magma or GAP record with information for the given refined passport (or several records representing all the refined passports corresponding to a specific group and signature). This should be the most useful part for researchers working on questions requiring computations of generating vectors, as these will no longer need to be computed from scratch. Also, the data can be searched over a variety of fields such as signature, or dimension of the family, or whether the family is hyperelliptic.

## 5. Future Work

We plan to add additional information to the database. Here are a few examples.

5.1. **Equivalence Relations.** Suppose we have two actions $\eta_1 : \Gamma \to G$ and $\eta_2 : \Gamma \to G$. We would like to determine when these are the "same" action analytically (conformally) or topologically. Two actions $\eta_1$ and $\eta_2$ are *conformally equivalent* if there is some $\omega \in \mathrm{Aut}(G)$ and $\widetilde{h} \in \mathrm{Aut}(\mathbb{H}) = \mathrm{PSL}(2, \mathbb{R})$ so that the following diagram commutes

$$\begin{array}{ccccc} K & \longrightarrow & \Gamma & \xrightarrow{\eta_1} & G \\ \big\downarrow{\scriptstyle \widetilde{h}^*} & & \big\downarrow{\scriptstyle \widetilde{h}^*} & & \big\downarrow{\scriptstyle \omega} \\ K & \longrightarrow & \Gamma & \xrightarrow{\eta_2} & G \end{array}$$

where $\widetilde{h}^*$ is the map that takes some $\gamma \in K$ or $\in \Gamma$ and sends it to $\widetilde{h}\gamma\widetilde{h}^{-1}$ [Broughton, 1991]. This definition induces a conformal mapping $h : X \to X$ where $X = \mathbb{H}/K$.

Two actions $\eta_1$ and $\eta_2$ are *topologically equivalent* if there exists an $\omega \in \mathrm{Aut}(G)$ and $\phi \in \mathrm{Aut}^+(\Gamma)$ so that the following diagram commutes [Broughton, 1991].

$$\begin{tikzcd} \Gamma \arrow[r, "\eta_1"] \arrow[d, "\phi"] & G \arrow[d, "\omega"] \\ \Gamma \arrow[r, "\eta_2"] & G \end{tikzcd}$$

Notice this means that $\eta_2 = \omega \circ \eta_1 \circ \phi^{-1}$ where $\phi$ is an element of $\mathrm{Aut}^+(\Gamma)$ and $\omega \in \mathrm{Aut}(G)$. As such, two actions are topologically equivalent precisely when they are in the same orbit under the action of $\mathrm{Aut}(G) \times \mathrm{Aut}^+(\Gamma)$ [Broughton, 1991, Proposition 2.2].

This translates the definition of topologically equivalent to an algebraic condition, and using this result for cases where $g_0 = 0$, code is implemented in Sage [Behn et al., 2015] which computes the orbits under this action, and returns one representative of each orbit. We plan to use their code to connect examples already in LMFDB which are topologically equivalent.

In the study of Hurwitz spaces (and the related inverse Galois problem) generating vectors up to the action of $\mathrm{Inn}(G) \times \mathrm{Aut}^+(\Gamma)$ are instead used. We also plan to collect together on each refined passport page generating vectors in the same orbit under this action.

5.2. **Quotient curves other than $\mathbb{P}^1$.** Currently all the data is for $g_0 = 0$. Breuer's data does include all possible cases where the quotient curve is higher genus, and the code written from algorithms in [Ries, 1993] will also determine if these actions describe the full automorphisms for a given family. However, there are a few small changes to the individual pages on LMFDB which will need to be made before including this data. For instance, the generating vector now must include $2g_0$ hyperbolic generators, and some of the additional information computed will be different for these coverings.

5.3. **Other topics.**

- David Swinarski and I plan to add additional information computed in [Swinarski, 2016]
- There is much current research on superelliptic curves, and we could incorporate known data about these families into LMFDB.
- The Riemann matrix and corresponding period matrix are crucial objects for understanding certain computational properties of Riemann surfaces.
- It would be nice to determine the fields of definition of these curves.

## 6. Acknowledgments

(in order of appearance)

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