

Math 3500 Spring 2011
Homework 2: Extra Problems
Due: January 28, 2011

1. Determine whether each of the following relations is reflexive, symmetric, and transitive (you should check each individual property, not all three at once). If a certain property fails, you should give a specific counterexample.
 - a. $A = \mathbb{Z}$ where $a \sim b$ means $a - b \neq 1$.
 - b. $A = \mathbb{Z}$ where $a \sim b$ means that both a and b are even.
 - c. $A = \mathbb{Z}$ where $a \sim b$ means $a \mid b$.

2. As in class, let $S = \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ and let \sim be the equivalence relation given by $(a, b) \sim (c, d)$ if $ad = bc$. Let Q be the set of equivalence classes of A under \sim .

- a. Show that if $(a_1, b_1) \sim (c_1, d_1)$ and $(a_2, b_2) \sim (c_2, d_2)$, then $(a_1a_2, b_1b_2) \sim (c_1c_2, d_1d_2)$.

- b. Show that if $(a_1, b_1) \sim (c_1, d_1)$ and $(a_2, b_2) \sim (c_2, d_2)$, then $(a_1b_2 + a_2b_1, b_1b_2) \sim (c_1d_2 + c_2d_1, d_1d_2)$.

The above parts show that the operations of addition and multiplication of fractions you learned in grade school are indeed well-defined on Q . In other words, the following definitions make sense:

$$\overline{(a, b)} \cdot \overline{(c, d)} = \overline{(ac, bd)} \quad \overline{(a, b)} + \overline{(c, d)} = \overline{(ad + bc, bd)}$$

- c. Let $a, b \in \mathbb{Z}$ with $b \neq 0$. Show that $\overline{(a, b)} + \overline{(-a, b)} = \overline{(0, 1)}$

- d. Let $a, b \in \mathbb{Z}$ with both $a, b \neq 0$. Show that $\overline{(a, b)} \cdot \overline{(b, a)} = \overline{(1, 1)}$.

Thus, every element has an additive inverse and every nonzero element (i.e. every element other than $\overline{(0, 1)}$) has a multiplicative inverse.

3. A friend tries to convince you that the reflexive property is redundant in the definition of an equivalence relation because they claim that symmetry and transitivity imply it. Here is the argument they propose:

“If $a \sim b$, then $b \sim a$ by symmetry, so $a \sim a$ by transitivity. This gives the reflexive property.”

Now you know that their argument must be wrong because one of the examples in Problem 1 is symmetric and transitive but not reflexive. Pinpoint the error in your friend’s argument. Be as explicit and descriptive as you can.