Math 215: Linear Algebra

Problem Set 12 : Due October 2

(28 points) Make sure you are familiar with the Academic Honesty policies for this class, as detailed on the syllabus. All work is due on the given day by 3 PM Grinnell Time, or 7 PM if you LaTeX the assignment. **For this assignment, show all of your computational work.**

1. Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation with $[T] = \begin{pmatrix} 2 & -1 \\ 5 & 2 \end{pmatrix}$. Let $\vec{u_1} = \binom{1}{3}$ and $\vec{u_2} = \binom{-2}{0}$ with $\alpha = (\vec{u_1}, \vec{u_2})$.
   (a) (2 points) Prove that $\text{Span}(\vec{u_1}, \vec{u_2}) = \mathbb{R}^2$.
   (b) (2 points) Compute $[T(\vec{u_1})]_\alpha$.
   (c) (2 points) Compute $[T(\vec{u_2})]_\alpha$.
   (d) (1 point) Use (b) and (c) to compute $[T]_\alpha$.

2. (2 points) Repeat problem (1) but use Proposition 3.4.7 to determine $[T]_\alpha$ where $\alpha = (\vec{u_1}, \vec{u_2})$.

3. (4 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation defined by $T\left(\binom{x}{y}\right) = \binom{rx}{ry}$ where $r \in \mathbb{R}$ is arbitrary. Prove that for all bases $\alpha = (\vec{u_1}, \vec{u_2})$ of $\mathbb{R}^2$ we have $[T] = [T]_\alpha$.

4. Suppose $[T] = \begin{pmatrix} 35 & 24 \\ -48 & -33 \end{pmatrix}$ and let $\alpha = \left(\binom{-2}{3}, \binom{-3}{4}\right)$ be a basis of $\mathbb{R}^2$.
   (a) (2 points) Compute $T(\vec{v})$ for **four** nonzero vectors $\vec{v} \in \text{Span}(\binom{-2}{3})$.
   (b) (2 points) Compute $T(\vec{v})$ for **four** nonzero vectors $\vec{v} \in \text{Span}(\binom{-3}{4})$.
   (c) (3 points) On one graph, carefully plot the eight vectors $\vec{v}$ from (a) and (b) and the eight vectors $T(\vec{v})$ for those eight vectors. (It is ok to plot them as points in $\mathbb{R}^2$ instead of as vectors.) Geometrically describe what the transformation is doing to these vectors.
   (d) (2 points) Use (c) to find values a and d so that $[T]_\alpha = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$. Explain briefly your reasoning.

5. (a) (3 points) Is the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ with $[T] = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}$ diagonalizable? If so, find some basis $\alpha = (\vec{u_1}, \vec{u_2})$ so that $[T]_\alpha$ is a diagonal matrix.
   (b) (3 points) Is the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ with $[T] = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$ diagonalizable? If so, find some basis $\alpha = (\vec{u_1}, \vec{u_2})$ so that $[T]_\alpha$ is a diagonal matrix.

6. (DO NOT TURN IN) Prove that for all 2×2 matrices A we have A is invertible if and only if 0 is not an eigenvalue of A. (Make sure you recall how to prove an *if and only if*.)