Math 215: Linear Algebra

PROBLEM SET 15 : DUE OCTOBER 12

(27 points) Make sure you are familiar with the Academic Honesty policies for this class, as detailed on the syllabus. All work is due on the given day by 3 PM Grinnell Time, or 7 PM if you LaTeX the assignment. **Make sure you describe all elementary row operations in the notation and manner discussed in class.**

1. (4 points) Determine for which values of a and $b \in \mathbb{R}$, the system with augmented matrix
$$\begin{pmatrix} 1 & 1 & 3 & 2 \\ 1 & 2 & 4 & 3 \\ 1 & 3 & a & b \end{pmatrix}$$
has (i) no solution (ii) one solution, and (iii) infinitely many solutions.

2. (4 points) Consider two systems
$$\begin{aligned} x_1 + 2x_2 + x_3 &= 2 \\ -x_1 - x_2 + 2x_3 &= 3 \\ 2x_1 + 3x_2 \qquad &= 0 \end{aligned}$$
and
$$\begin{aligned} x_1 + 2x_2 + x_3 &= -1 \\ -x_1 - x_2 + 2x_3 &= 2 \\ 2x_1 + 3x_2 \qquad &= -2. \end{aligned}$$
Notice that the left sides of these two systems are identical. Instead of solving these two problems separately, combine them into one "super augmented" matrix and use our technique to find solution(s) to both systems all at once.

3. (a) (3 points) Let $V = \mathbb{R}^4$. Does $\text{Span}\left(\begin{pmatrix} 1 \\ 3 \\ -3 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 4 \\ -1 \\ 1 \end{pmatrix}\right) = \mathbb{R}^4$? Carefully prove this if it is true. Give an explicit element of V which is not in the span otherwise.

   (b) (2 points) How do we know without doing any computations that $\text{Span}\left(\begin{pmatrix} 1 \\ 3 \\ -2 \\ 5 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 4 \\ -6 \end{pmatrix}\right)$ is not all of $\mathbb{R}^4$?

4. (4 points) Let $V = P_3$. Does $\text{Span}(x^2 - 1, x + 3, x^3 + x^2 + x, x^3) = V$? Carefully prove this if it is true. Give an explicit element of V which is not in the span otherwise.

5. (2 points) Let $V = \mathcal{F}$ be the vector space of all functions from $\mathbb{R}$ to $\mathbb{R}$. Consider the vectors $f(x) = \sin^2(x)$, $g(x) = \cos^2(x)$, and $h(x) = -3$. Is $(f(x), g(x), h(x))$ linearly independent or linearly dependent? Carefully explain your answer.

6. (4 points) Use the process of row reduction (also called "Gaussian Elimination") to determine if the following set of vectors in $\mathbb{R}^4$ are linearly independent.

$$\left(\begin{pmatrix} 1 \\ 3 \\ -2 \\ 5 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 4 \\ -6 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ -1 \\ 1 \end{pmatrix} \right)$$

7. (4 points) Let W be the set of elements in $\mathbb{R}^4$ all of whose vector components are the same. So $\begin{pmatrix} 4 \\ 4 \\ 4 \\ 4 \end{pmatrix}$ is an element of W. This set W is a subspace of $\mathbb{R}^4$ (it's a good exercise to prove this on your own). Determine the dimension of W.

8. (DO NOT TURN IN.) Let $V = \mathcal{M}_{2\times 2}$. Does $\text{Span}\left(\begin{pmatrix} 1 & 1 \\ 0 & -3 \end{pmatrix}, \begin{pmatrix} 2 & -4 \\ 0 & 5 \end{pmatrix} \right) = V$? Explain why or why not.

9. (DO NOT TURN IN.) Let $V = P_4$ be the vector space of all polynomials of degree at most 4, plus the zero polynomial. Consider the vectors $(x^4 + x^3, 3x^4 + 2x^2, 3x^3 - x^2)$. Is this list of vectors linearly independent or linearly dependent? Carefully explain your answer.