Math 215: Linear Algebra

Problem Set 16 : Due October 13

(27 points) Make sure you are familiar with the Academic Honesty policies for this class, as detailed on the syllabus. All work is due on the given day by 3 PM Grinnell Time, or 7 PM if you LaTeX the assignment. Show all calculations. **Make sure you describe all elementary row operations in the notation and manner discussed in class.**

1. (5 points) The following list of vectors span $\mathbb{R}^3$:

$$\left(\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 7 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right).$$

   Find a basis of $\mathbb{R}^3$ consisting of (some of) these vectors.

2. (5 points) The two vectors $\left(\begin{pmatrix} 1 \\ 3 \\ 4 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \\ 3 \end{pmatrix} \right)$ are linearly independent. Find a basis of $\mathbb{R}^4$ which contains these two vectors.

3. Below are lists of elements in P_3. The span of these vectors is a subspace by Prop. 4.1.16. For each part, determine the dimension of the subspace of P_3 which is the span of these elements.
   (a) (3 points) $(x^2, x^2 - x - 1, x + 1)$
   (b) (3 points) $(x, x - 1, x^2 + 1)$

4. (5 points) The **transpose** of a matrix is the matrix which is formed by swapping rows and columns of a matrix. We can interpret the transpose as a linear transformation $T : \mathcal{M}_{2\times 2} \to \mathcal{M}_{2\times 2}$ where $T\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$. If we let $\alpha = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right)$ a basis of $\mathcal{M}_{2\times 2}$, what is $[T]_\alpha^\alpha$? Show all your computations and briefly describe what you are doing.

5. Define a linear transformation $T : P_2 \to \mathbb{R}^2$ as $T(f(x)) = \begin{pmatrix} f(3) \\ f(0) \end{pmatrix}$. Let $\alpha = (x^2, x, 1)$ be a basis of P_2, ε_2 the standard basis of $\mathbb{R}^2$, and $\beta = \left(\binom{1}{-1}, \binom{2}{1} \right)$ another basis of $\mathbb{R}^2$.

(a) (3 points) What is $[T]_\alpha^{\varepsilon_2}$? Show your computations.

(b) (3 points) What is $[T]_\alpha^\beta$? Show your computations.

6. (DO NOT TURN IN.)

   (a) Is it possible to find a pair of two-dimensional subspaces U and V of $\mathbb{R}^3$ such that $U \cap V = \{\vec{0}\}$? Prove or disprove your answer. (Suggestion: Suppose $(\vec{u_1}, \vec{u_2})$ is a basis of U and $(\vec{v_1}, \vec{v_2})$ is a basis of V. What can you say about the sequence $(\vec{u_1}, \vec{u_2}, \vec{v_1}, \vec{v_2})$?)

   (b) Give a geometrical interpretation of your conclusion from (a). (You might want to think about what two-dimensional subspaces of $\mathbb{R}^3$ look like.)

7. (DO NOT TURN IN.) Let $V = P$, the set of all polynomials with $p(x) \in P$. Which of the following are linear transformations $T : P \to P$? For those that are, carefully prove they are. For those that are not, carefully explain what property fails to be true.

   (a) $T(p(x)) = p(x^2)$

   (b) $T(p(x)) = (p(x))^2$

   (c) $T(p(x)) = x^2 p(x)$